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A METHOD FOR THE SOLUTION OF THE TWO-DIMENSIONAL MULTIFRONT STEFAN PROBLEM

A. S. Ovcharova

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A method for the solution of the thermoconvective problem in a multilayer medium with curvilinear and, in the general case, mobile interfaces is proposed. Such a situation can occur when temperature fields in a multilayer medium (for example, oil, ground, and water) with different thermal conductivity of the layers must be calculated, or in problems connected with crystallization (melting). The idea of the method is based on the decomposition of a complex system into unitary elements (modules) given that there is a set of rules for their coupling with each other. Each module is a single-type local model where the calculation is organized independently of the other modules of the system, while the set of rules determining, for example, the common boundaries between the modules and their functional dependences on them, combines these modules into the initial system. The problem is solved under the following assumptions: 1) the thermophysical properties of the substances in each medium are constant; 2) the density jump during the phase transition is ignored.

1. The Mathematical Model. Let a domain G with solid sides $0 \leqslant x \leqslant 1$, $f_0(x) \leqslant y \leqslant f_M(x)$ include M substances separated by curvilinear and, in the general case, mobile boundaries $f_m(t,x)$ $(m = 1,\ldots,M-1)$ (Fig. 1). We shall assume that in each of the subdomains filled by the liquid phase of the substance, convective movement of the liquid takes place. In the Oberbeck–Boussinesq approximation [1], it is described by a system of equations, which can be represented in dimensionless form, as in [2]:

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}\left(\omega \frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\omega \frac{\partial \psi}{\partial x}\right) = \Delta\omega + \mathrm{Gr}\frac{\partial T}{\partial x}; \tag{1.1}$$

$$\Delta\psi + \omega = 0; \tag{1.2}$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}\left(T\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(T\frac{\partial \psi}{\partial x}\right) = \frac{1}{\mathrm{Pr}}\Delta T. \tag{1.3}$$

In each of the subdomains filled by the solid phase, the equation of thermal conductivity is fulfilled:

$$\frac{\partial T}{\partial t} = \frac{1}{\mathrm{Pr}}\Delta T. \tag{1.4}$$

Here $\mathrm{Gr} = g\beta\theta x_0^3/\nu^2$ is the Grashof number (x_0 is the size in x); $\mathrm{Pr} = \nu/\chi$ is the Prandtl number; g is the acceleration of gravity (the vector of gravity is parallel to the y axis and is pointing downward); the stream function ψ is introduced by the relations

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}. \tag{1.5}$$

The sought-for functions are the velocity vortex $\omega(t,x,y)$, the stream function $\psi(t,x,y)$, the temperature $T(t,x,y)$, as well as the interfaces $f_m(t,x)$ if crystallization or melting take place. We shall consider that the initial conditions for the functions ω, ψ, T, f_m are specified.

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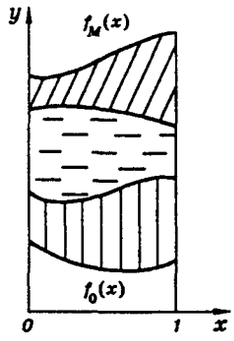

Fig. 1

The boundary conditions for the temperature on the external sides of the domain G ($x = 0$, $x = 1$, $y = f_0(x)$, $y = f_M(x)$) can be given in the form

$$\alpha T + \beta \frac{\partial T}{\partial n} = \gamma(t) \tag{1.6}$$

(α and β take the values 0 and 1).

At all the internal boundaries, i.e., on the curves $f_m(t,x)$ ($m = 1,\dots,M-1$), the conditions of conjugation are fulfilled:

$$T^m(x, f_m(t,x)) = T^{m+1}(x, f_m(t,x)); \tag{1.7}$$

$$V_m^n = k_m \frac{\partial T^m}{\partial n} - k_{m+1} \frac{\partial T^{m+1}}{\partial n}, \tag{1.8}$$

where n is the vector of the normal to the curve $f_m(t,x)$ at the point x; k_m, T^m and k_{m+1}, T^{m+1} are dimensionless coefficients of thermal conductivity and the temperature for the substances filling the subdomains m and $m+1$, respectively; V_m^n is the movement speed of the boundary $f_m(t,x)$ in the direction of the normal to it. Hereafter the index m (superscript or subscript) denotes that the function or operator belongs to the subdomain m bounded by the curve f_{m+1} from above and by the curve f_m ($m = 0,\dots,M-1$) from below.

The boundary conditions for the stream function and the velocity vortex are given in the form of viscous nonslip conditions, i.e.,

$$\mathrm{v}|_\Gamma = 0 \tag{1.9}$$

(Γ is the boundary of the subdomain m).

2. Method of Solution. First of all, note that Eqs. (1.1)–(1.4) are of the same type within the framework of the Oberbeck–Boussinesq model. This statement also applies to Eq. (1.2), which will be solved by the iteration method and can be brought to the general type by introducing an iterative parameter.

Let us map each of the subdomains m of the form $0 \leqslant x \leqslant 1$, $f_m(t,x) \leqslant y \leqslant f_{m+1}(t,x)$ ($m = 0,\dots,M-1$) into a square with sides $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$ using the transformation

$$\xi = x, \quad \eta = \frac{y - f_m(t,x)}{f_{m+1}(t,x) - f_m(t,x)}. \tag{2.1}$$

Then each of Eqs. (1.1)–(1.4) can be represented in the form

$$\frac{\partial \Phi}{\partial t} = \frac{1}{B_m I_m}\left[\frac{\partial}{\partial \xi}\left(B_{11}^m \frac{\partial \Phi}{\partial \xi} + B_{12}^m \frac{\partial \Phi}{\partial \eta} - A_1^m \Phi \frac{\partial \psi}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(B_{12}^m \frac{\partial \Phi}{\partial \xi} + B_{22}^m \frac{\partial \Phi}{\partial \eta} + A_1^m \Phi \frac{\partial \psi}{\partial \xi}\right)\right] + R_1^m \frac{\partial \Phi}{\partial \eta} + R_2^m. \tag{2.2}$$

Here

$$I_m = f_{m+1}(t,\xi) - f_m(t,\xi), \quad R_1^m = \frac{1}{I_m}\left(\frac{\partial I_m}{\partial t}\eta + \frac{\partial f_m}{\partial t}\right),$$

$$B_{11}^m = I_m, \quad B_{12}^m = -\Big(\frac{\partial I_m}{\partial \xi}\eta + \frac{\partial f_m}{\partial \xi}\Big), \quad B_{22}^m = \frac{1 + (B_{12}^m)^2}{I_m}. \tag{2.3}$$

Hence it follows that

$$B_{11}^m B_{22}^m - (B_{12}^m)^2 = 1. \tag{2.4}$$

If a subdomain filled by a liquid phase is considered, then we have for $\Phi = \omega$

$$B_m = 1, \quad A_1^m = 1, \quad R_2^m = \frac{\mathrm{Gr}_m}{I_m}\Big(B_{11}^m \frac{\partial T}{\partial \xi} + B_{12}^m \frac{\partial T}{\partial \eta}\Big),$$

for $\Phi = T$

$$B_m = A_1^m = \mathrm{Pr}_m,$$

and for $\Phi = \psi$

$$B_m = \frac{1}{\lambda}, \quad A_1^m = 0, \quad R_1^m = 0, \quad R_2 = \omega\lambda$$

(Gr_m, Pr_m are the Grashof and Prandtl numbers, λ is an iteration parameter).

If a subdomain filled by a solid phase is considered, then we have for $\Phi = T$

$$B_m = \mathrm{Pr}_m, \quad A_1^m = 0, \quad R_2^m = 0.$$

Let us introduce the following notations:

$$U(\Phi) = B_{11}\frac{\partial \Phi}{\partial \xi} + B_{12}\frac{\partial \Phi}{\partial \eta} - A_1\Phi\frac{\partial \psi}{\partial \eta}, \qquad V(\Phi) = B_{12}\frac{\partial \Phi}{\partial \xi} + B_{22}\frac{\partial \Phi}{\partial \eta} + A_1\Phi\frac{\partial \psi}{\partial \xi}, \tag{2.5}$$

and, using relations (2.4), we have [3]

$$U(\Phi) = \frac{1}{B_{22}}[\Phi_\xi + B_{12}V(\Phi) - B_1A_1\Phi], \quad V(\Phi) = \frac{1}{B_{11}}[\Phi_\eta + B_{12}U(\Phi) + B_2A_1\Phi], \tag{2.6}$$

where

$$B_1 = B_{12}\frac{\partial \psi}{\partial \xi} + B_{22}\frac{\partial \psi}{\partial \eta}; \quad B_2 = B_{11}\frac{\partial \psi}{\partial \xi} + B_{12}\frac{\partial \psi}{\partial \eta}.$$

Then, for each m, Eq. (2.2) has the form

$$\frac{\partial \Phi}{\partial t} = \frac{1}{BI}\Big[U_\xi(\Phi) + V_\eta(\Phi)\Big] + R_1\Phi_\eta + R_2. \tag{2.7}$$

Equation (2.7) is obtained in the form of conservation laws, which is especially important for calculation of temperature.

The choice of mathematical model and method of solution extended all the difficulties associated with the solution of the problem as a whole to the solution of a parabolic equation in a "good" domain, but having variable coefficients and mixed derivatives. Here the coefficients of the differential equation depend eventually on the solution itself. In solving this type of problem, the method of decomposition into physical processes [4] which lead to the problem splitting into evolutional and boundary problems gives good results. In the evolutional problem, the solution to Eq. (2.7) with the coefficients $B_{11}^s(t_k)$, $B_{12}^s(t_k)$, $B_{22}^s(t_k)$ taken from the previous iteration step s is found at each time step with the help of an appropriate difference scheme. In the boundary problem, a matrix for the coefficients of Eq. (2.7) is determined, that is, $B_{11}^s(t_k)$, $B_{12}^s(t_k)$, $B_{22}^s(t_k)$, using the solution to this equation. The solution obtained at the previous time step is taken as an initial iteration.

The solution to Eq. (2.7) at each time step will be sought using the stabilizing correction scheme [5] taken in the form

$$\frac{\Phi^{k+1/2}-\Phi^k}{\tau}=\frac{1}{BI}\Big[U_\xi^k(\Phi)+V_\eta^{k+1/2}(\Phi)\Big]+R_1\Phi_\eta^{k+1/2}+R_2^{k+1/2},$$
$$\frac{\Phi^{k+1}-\Phi^{k+1/2}}{\tau}=\frac{1}{BI}\Big[U_\xi^{k+1}(\Phi)-U_\xi^k(\Phi)\Big]. \tag{2.8}$$

Here

$$V(\Phi)^{k+1/2}=\frac{1}{B_{11}}\Big[\Phi_\eta^{k+1/2}+B_{12}U^k(\Phi)+B_2A_1\Phi^{k+1/2}\Big],$$
$$U^{k+1}(\Phi)=\frac{1}{B_{22}}\Big[\Phi_\xi^{k+1}+B_{12}V^{k+1/2}(\Phi)-B_1A_1\Phi^{k+1}\Big]. \tag{2.9}$$

The stabilizing correction scheme falls into the class of economical difference schemes with fractional steps, where the first fractional step gives a full approximation of the equation and the next step is corrective and serves to improve of stability.

In order to realize the scheme (2.8), (2.9) in each of the squares corresponding to the subdomains m, a rectangular grid is constructed in a standard way:

$$\xi_i=(i-1)\,\Delta\xi,\quad \Delta\xi=1/L,\quad i=1,\dots,LL,\quad LL=L+1,$$
$$\eta_j=(j-1)\Delta\eta,\quad \Delta\eta=1/J,\quad j=1,\dots,JJ,\quad JJ=J+1.$$

Differential expressions of the type $(a_{11}\Phi_\xi)_\xi$, $(a_{22}\Phi_\eta)_\eta$, Φ_ξ, Φ_η are approximated, with the second order of accuracy, by the finite-difference analogs $\Lambda_{11},\Lambda_{22},\Lambda_1,\Lambda_2$, respectively, which have traditional representations [5, 6]. For the approximation of the mixed derivative, for example, $(a_{12}\Phi_\xi)_\eta$, we used, in accordance with [5, 7], the operator

$$\Lambda_{12}\Phi=\frac{(a_{12})_{i,j+1}(\Phi_{i+1,j+1}-\Phi_{i+1,j-1})-(a_{12})_{i,j-1}(\Phi_{i-1,j+1}-\Phi_{i-1,j-1})}{4\,\Delta\xi\,\Delta\eta}.$$

The operator $\Lambda_{21}\Phi$ is determined in a similar way. Then scheme (2.8), (2.9) will approximate (2.7) with an accuracy $O(\tau+h^2)$, taking into account boundary conditions similar to (1.6). It is absolutely stable and converges to the solution of (2.7) at $\tau\to 0$ [5]. It should be noted that the representations for $U(\Phi)$ and $V(\Phi)$ taken in the form of (2.6) enhance the stability of the scheme at $B_{12}>1$, in contrast to (2.5), owing to the coefficient $1/B_{22}$ [3].

Replacing the derivatives in (2.8) by the corresponding finite differences and substituting the difference analogs of $V_\eta^{k+1/2}(\Phi)$ and $U_\xi^{k+1}(\Phi)$ from (2.9), we obtain a system of linear difference equations relative to the function $\Phi(\xi_i,\eta_j)$ at each half-time step $(k+1/2,\,k+1)$ for all internal points $(i=2,\dots,L,\,j=2,\dots,J)$. The system has a three-diagonal structure with diagonal elements predominating in the matrix, and can be effectively solved by the Thomas algorithm, taking into account specific boundary conditions.

The iterative process is realized in the following way. Let the solution to Eq. (2.7) for the temperature and location of the phase boundaries $f_m(t_k,x)$ at the time $t=t_k$ be known. On determining the movement velocity of each of the boundaries $V_m^n(t_{k+1},x)$ in the directions normal to these boundaries using the Stefan condition, we find new locations of the phase boundaries $f_m(t_{k+1},x)$ at the time $t_{k+1}=t_k+\tau$ (Fig. 2) for each m. Then, the matrix of the coefficients B_{11}, B_{12}, B_{22} of Eq. (2.7) is determined using formulas (2.3).

Finally, after solving this equation with new coefficients, we obtain the temperature distribution in each of the subdomains m at the time t_{k+1} at the iteration $s=0$. At each time step, iterations are performed until the condition

$$\max_{m,i}\frac{|(V_m^n(x_i))^{s+1}|-|(V_m^n(x_i))^s|}{|(V_m^n(x_i))^{s+1}|}<\varepsilon$$

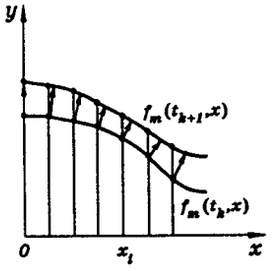

Fig. 2

is fulfilled. Here s is the iteration number, and ε is the given accuracy.

As is obvious from the description, the iterative process organized in this way does not violate the conservation laws in any of the subdomains m, and the approximating operator retains its elliptical form at each iteration in any of the subdomains m. It is these properties that provide convergence of the iterative process [4, 8].

3. Calculation of Temperature. Let us transform the boundary conditions (1.6)–(1.8) to new variables. Then condition (1.6) relating to the external solid boundaries of the domain G will take the following form: for each subdomain m at $\xi = 0$ and $\xi = 1$,

$$\alpha_m T^m \Big|_{\substack{\xi=0 \\ \xi=1}} + \frac{\beta_m}{I_m} U(T^m) \Big|_{\substack{\xi=0 \\ \xi=1}} = \gamma_m(t) \tag{3.1}$$

(α, β, γ take their values for each of the sides $\xi = 0$, $\xi = 1$); for the subdomain $m = 1$ and $\eta = 0$

$$\alpha_1 T^1|_{\eta=0} + \frac{\beta_1}{\sqrt{1 + (B_{12}^1)^2}\Big|_{\eta=0}} V(T^1)|_{\eta=0} = \gamma_1(t); \tag{3.2}$$

for the subdomain $m = M$ and $\eta = 1$

$$\alpha_M T^M|_{\eta=1} + \frac{\beta_M}{\sqrt{1 + (B_{12}^M)^2}\Big|_{\eta=1}} V(T^M)|_{\eta=1} = \gamma_M(t). \tag{3.3}$$

At the internal boundaries, conjugation conditions (1.7) and (1.8) will take, respectively, the following form:

$$T^m(\xi, 1) = T^{m+1}(\xi, 0), \quad m = 1, \ldots, M-1; \tag{3.4}$$

$$V_m^n = \frac{1}{\sqrt{1 + (B_{12}^m)^2|_{\eta=1}}} [k_m V(T^m)|_{\eta=1} - k_{m+1} V(T^{m+1})|_{\eta=0}]. \tag{3.5}$$

In the calculation of the temperature field of each of the subdomains m from the domain G, three cases are possible.

A. All the boundaries f_m are fixed and known in advance, and the temperature at the interface is unknown. The following scheme of solution is proposed for determination of the temperature field in the domain G. At the first stage, the first equation of system (2.8) in the direction η is solved. Then we obtain the following system of equations for all the internal points ($i = 2, \ldots, L$, $j = 2, \ldots, J$) of each of the subdomains m:

$$-a_j^m T_{j-1}^m + b_j^m T_j^m - c_j^m T_{j+1}^m = d_j^m. \tag{3.6}$$

Hereafter the indices i are omitted for simplicity. The solution to system (3.6) will be sought using the

parametric Thomas algorithm [9]

$$T_j^m = P^m(j)T_*^m + Q^m(j)T_{**}^m + R^m(j), \quad m = 1, \ldots, M, \tag{3.7}$$

where P, Q, R are one-dimensional arrays; T_*^m is the temperature at the lower boundary of the subdomain m; T_{**}^m is the temperature at its upper boundary. In order to determine the unknowns T_*^m and T_{**}^m, we use the conditions of conjugation (3.5), assuming that $V_m^n = 0$:

$$k^m\left[B_{12}^m\frac{\partial T^m}{\partial \xi} + B_{22}^m\frac{\partial T^m}{\partial \eta}\right]\Big|_{\eta=1} = k^{m+1}\left[B_{12}^{m+1}\frac{\partial T^{m+1}}{\partial \xi} + B_{22}^{m+1}\frac{\partial T^{m+1}}{\partial \eta}\right]\Big|_{\eta=0}. \tag{3.8}$$

Substituting, instead of the temperature values at the near-boundary points, their expressions using (3.7) into the difference analog of this equation and taking into account that $T_{**}^m = T_*^{m+1}$, we obtain, in order to determine T_*^m, a system of m equations with a three-diagonal structure of the matrix [10]. At large M, it can be solved by the Thomas algorithm, taking into account the boundary conditions (3.2) and (3.3). Then, using formulas (3.7), we reconstruct all the other temperature values for each m.

At the second stage, the second equation of (2.8) for each m in the direction ξ is solved. The systems of difference equations obtained are solved by the conventional Thomas algorithm, taking into account boundary conditions (3.1).

B. All the internal boundaries $f_m(t, x)$ are mobile. In this case, the temperature at the interface is known [temperature of crystallization (melting)]. The main difficulty is that in the calculation of temperature, the coefficients of difference equations [see (3.6)] depend on locations of the boundaries $f_m(t, x)$, which, in turn, depend on temperature. Therefore, system of equations (2.8) for determination of temperature is solved together with condition (3.5) using the iterative process described in Sec. 2 for all m. The value from the lower time layer is taken as an initial approximation of V_m^n. The systems of difference equations obtained are solved by the Thomas algorithm.

C. Only a part of the internal boundaries f_m is mobile. In this case, the system of equations (2.8) for the determination of temperature is solved together with (3.4), (3.5) by an iterative method for V_m^n. The systems of difference equations obtained are solved by the parametric Thomas algorithm using the condition that the temperature at the mobile boundaries is known.

4. Calculations of Velocity Vortex and Stream Function. In the solution of the Navier–Stokes equations using the variables (ψ, ω), the boundary conditions for ω are not specified explicitly, as a rule. Both boundary conditions are given for ψ.

Further, one of these conditions is used to solve the equation that determines the stream function, and the other is used to obtain expressions relating the stream function to the vortex at the boundary. The formulas of Thoma, Woods, etc. are examples of such relations.

Let a subdomain m be bounded from below by the curve $f_m(t, x)$ (which corresponds to $\eta = 0$), and from above — by the curve $f_{m+1}(t, x)$ $(\eta = 1)$. We need not give our reasoning for both boundaries, but only for one boundary, say for $f_m(t, x)$. From relations (2.5) for the stream function and (1.9) it follows that, at $\eta = 0$,

$$U(\psi)\Big|_{\eta=0} = 0, \quad V(\psi)\Big|_{\eta=0} = 0. \tag{4.1}$$

In order to derive an expression relating the velocity vortex to the stream function at the boundary, let us turn to Eq. (1.2), assuming that it is valid everywhere, up to the boundary. With the new variables (ξ, η), (1.2) will take the form

$$\omega = -\frac{1}{I_{m+1}}\left[\frac{\partial U(\psi)}{\partial \xi} + \frac{\partial V(\psi)}{\partial \eta}\right] \tag{4.2}$$

with conditions (4.1) at $\eta = 0$.

If $V_{i,2}(\psi)$ is expanded into a Taylor series in the vicinity of the point $(i,1)$ lying on the line $\eta = 0$ in the direction η, we obtain

$$V_{i,2}(\psi) = V_{i,1}(\psi) + \left.\frac{\partial V(\psi)}{\partial \eta}\right|_{i,1} \Delta\eta + O(\Delta\eta^2). \tag{4.3}$$

Then (4.2), together with (4.1) and (4.3), gives a condition for ω which is analogous to the Thoma condition on the line $\eta = 0$ (the corresponding mobile or fixed boundary f_m):

$$\omega_{i,1} = -\frac{1}{I_m}\left[\frac{V_{i,2}(\psi)}{\Delta\eta}\right]. \tag{4.4}$$

Actually, if the subdomain m is a rectangle, then

$$\omega_{i,1} = -\frac{1}{\Delta\eta}\left(\frac{\partial\psi}{\partial\eta}\right)_{i,2} = -\frac{\psi_{i,3}}{2(\Delta\eta)^2}.$$

On the other hand, using the three-point approximation of the second order of accuracy for the derivative $\partial\psi/\partial\eta = 0$ at $\eta = 0$ and $\psi = 0$, we obtain [11] $\psi_{i,3} = 4\psi_{i,2}$. Hence, we obtain the classical Thoma condition

$$\omega|_{i,1} = -\frac{2}{(\Delta\eta)^2}\psi_{i,2}.$$

For the calculation of the velocity vortex and stream function, the following scheme is proposed. At the first stage, Eq. (2.7) for the velocity vortex is solved using formulas (2.8). The boundary condition for it is taken from the lower time layer. At the second stage, the equation for the stream function is solved by iteration using the same formulas with boundary conditions (4.1). In order to relate the velocity vortex to the stream function, formula (4.4) is used. The iterative process at each time step is considered to be completed when the condition

$$\max_{i,j} \frac{|\omega_{i,j}^{s+1} - \omega_{i,j}^{s}|}{|\omega_{i,j}^{s+1}|} < \varepsilon$$

is achieved.

Thus, one time step calculation of the velocity vortex and the stream function is finished. Then the process is repeated.

Remark. The Stefan problem with convection is often investigated with the help of the shock capturing method for the equation of thermal conductivity with discontinuous coefficients, as proposed in [12]. As is shown in [2], the main difficulty in this case is associated with approximation of the functions being calculated near the crystallization front. When the smoothed heat capacity is used, pulsations of temperature occur in the area of the interface. They automatically cause pulsations of the stream function. In order to obtain a smooth solution, weight coefficients were introduced for the temperature, and special approximations rearranging themselves with the movement of the crystallization front were used for the stream function.

Numerical calculations by the method proposed in this paper, using discontinuity, fitting and exactly satisfying the conditions on this discontinuity, showed that this method provides sufficient smoothness of the solution in each of the subdomains m. As for the program realization of the method for the calculations with the help of a personal computer, it is rather simple, since all the sought-for functions (T, ω, ψ) are calculated using the same computational procedure, taking into account boundary conditions. The main advantage of the method proposed is that it can help to effectively investigate the Stefan problem for thermal diffusion, i.e., crystallization (or melting) of a binary mixture, when the crystallization temperature of the melt is not a constant known value, but some function of the concentration of an admixture component in the melt. The investigation of the Stefan problem for thermal diffusion with the help of the method of heat capacity smoothing is rather problematical.

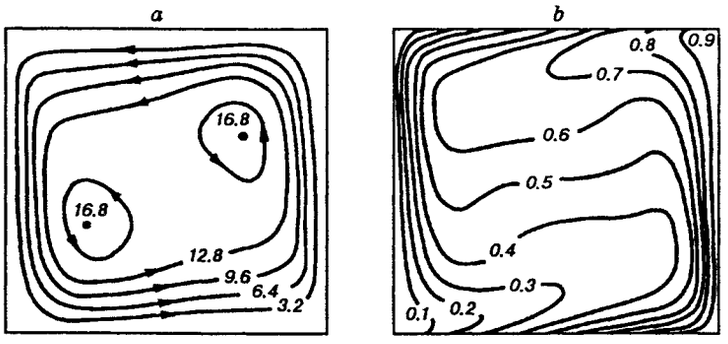

Fig. 3

Examples of calculation. In the examples considered below, calculations in each of the subdomains were carried out using grids of size 21 × 21 for the Grashof number Gr = 200,000 and the Prandtl number Pr = 1. The thermophysical characteristics of water and ice were used as thermophysical characteristics for the liquid and solid phases in the calculations with phase transitions.

1. In order to check the accuracy and effectiveness of the numerical algorithm, it is necessary to use exact solutions of the model equations or well-studied approximated solutions. Calculation of convective heat transport with lateral heating is a typical and well-studied example for testing a numerical algorithm in the solution of problems involving thermal convection [2].

Equations (1.1)–(1.3) are solved in a square closed domain with fixed boundaries. The velocity, the stream function, and the normal derivative of the stream function are equal to zero at the boundary of the domain. The temperature of the lateral sides is constant:

$$T(t,0,y) = 0, \qquad T(t,1,y) = 1.$$

On the horizontal sides, a linear distribution of temperature is specified:

$$T(t,x,y) = x.$$

The test consists in obtaining the stationary solution until the following inequality is satisfied:

$$(E^{k+1} - E^k)/E^k + (\mathrm{Nu}^{k+1} - \mathrm{Nu}^k)/\mathrm{Nu}^k < \varepsilon.$$

Here E is kinetic energy; Nu is the net thermal flow (the Nusselt number); ε is the given error; k is the number of time steps.

Results of the calculation by the method proposed are given in Fig. 3, where *a* stands for isolines of the stream function and *b* stands for isotherms. Comparison of the calculation results with the data presented in [2] showed high accuracy of the calculation by the method proposed. It is remarkable that the use of formula (4.4) relating the stream function to the velocity vortex at the boundary gives a more accurate result for the stream function (we mean $\max\psi$) than the use of the classical Thoma condition. The qualitative picture of the current fully coincides with that given in [2].

2. At the initial time, the domain G is a rectangle with sides $x = 0$, $x = 1$, $y = 0$, $y = 2$ filled with a substance in the liquid phase (subdomain I) and solid phase (subdomain II) with a curvilinear interface between them $f(0,x) = 1 - 0.1\sin\pi x$ (Fig. 4) and the following temperature distribution:

$$y = 0\colon\ T = 1; \quad f(0,x)\colon\ T = 0; \quad y = 2\colon\ T = -0.01.$$

At all the internal points of subdomains I and II, the temperature is specified by a linear law. At the initial time, the liquid is at rest. The boundary conditions have the form

$$x = 0\colon\ \frac{\partial T}{\partial n} = 0; \quad x = 1\colon\ \frac{\partial T}{\partial n} = 0;$$
$$y = 0\colon\ T = 1; \quad y = 2\colon\ T = -0.01.$$

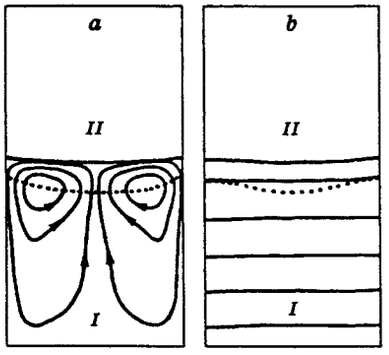

Fig. 4

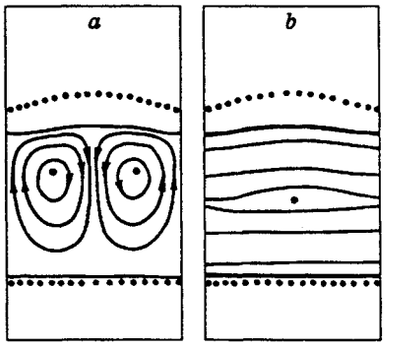

Fig. 5

Figure 4 presents the initial location of the interface (dotted line), the new location of the interface (solid line), isolines of the stream function (a), and isotherms (b) for the time $t = 1.19$. The convective movement occurring in the liquid phase as a result of the boundary curvature forms two vortices. The stream lines are directed through the center from the bottom upwards. The vortex movement tends to straighten out the interface.

3. The domain $G(0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 3)$ is filled with a substance in the liquid (in the middle) and solid (above and below) phases with interfaces $f_1(0,x) = 1$, $f_2(0,x) = 2 + 0.1 \sin \pi x$ between them with the following initial temperature distribution:

at the fixed boundaries

$$y = 0 \quad \text{and} \quad y = 3: \quad T = -0.01,$$

at the interfaces

$$f_1 \quad \text{and} \quad f_2: \quad T = 0.$$

In the solid phase, a linear distribution of temperature is given. In the liquid phase, the temperature increases from 0 to 1 from the boundary f_1 to the middle of the subdomain; then, from the middle of the subdomain to its boundary f_2, the temperature decreases from 1 to 0. At initial time the liquid is at rest. The boundary conditions are as follows:

$$x = 0, \quad x = 1: \quad \frac{\partial T}{\partial n} = 0; \quad y = 0: \quad T = -0.01.$$

At $y = 3$, the temperature is initially constant ($T = -0.01$) until the development of a convective flow ($t = 0.14$). Then it abruptly decreases to $T = -1.5$ and further is kept constant.

Figure 5 presents the initial location of the interfaces (dotted lines), the new location (solid lines), isolines of the stream function (a), and isotherms (b) for the time $t = 0.49$. The stream lines are directed through the center of the domain from top to bottom. The formed vortex movement is "drawing in" the upper boundary of the interface and slightly bends the lower boundary.

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